

Note on the perihelion/periastron advance due to cosmological constant

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Abstract We will comment on the perihelion/periastron advance of celestial bodies due to the cosmological constant Λ . It is well known that the cosmological constant Λ causes the perihelion/periastron shift; however, there seems to still exist a discrepancy among the various derived precession formulae. We will point out that the expression $\Delta\omega_\Lambda = (\pi c^2 \Lambda a^3 / (GM)) \sqrt{1-e^2}$ is the general formula for any orbital eccentricity e and the expression $\Delta\omega_\Lambda = (\pi c^2 \Lambda a^3 / (GM))(1-e^2)^3$ comes from the nearly circular ($e \ll 1$) approximation.

Keywords Celestial Mechanics · Gravitation · Cosmological Constant/Dark Energy

1 Introduction

The 2011 Nobel Prize in physics was awarded for the discovery of the accelerating expansion of the Universe [1,2]. According to the present literature, theory that is most suitable for explaining this phenomenon is that of the cosmological constant Λ , generally referred to as dark energy. Although the cosmological constant or dark energy can be considered to account for the current expansion of the Universe in the simple terms, the details are still far from clear. Meanwhile, several attempts have also been made for explaining this accelerating expansion without dark energy [3,4,5,6,7,8,9,10].

To verify the existence of the cosmological constant/dark energy and/or the potentiality of alternative gravitational theories, it is most natural to investigate their properties in

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the context of classical tests of general relativity, i.e., the perihelion/periastron advance of celestial bodies and the deflection of light. To this end, we must elucidate the difference in such effects among the cosmological constant and the alternative gravitational theories. Therefore, it is important to derive a rigorous formula for such effects resulting from the cosmological constant Λ . Recently, the effect of the cosmological constant on the bending of light rays was intensively investigated and discussed by many authors; the detailed reports can be found in [11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and the references therein.

The influence of the cosmological constant on the orbital motion is a historical problem (see section 45 of [21]). In principle, it is clear that the cosmological constant contributes to the perihelion/periastron advance, although this contribution is too small to detect because of $\Lambda \approx 10^{-52} \text{ m}^{-2}$. However, there is a discrepancy in the obtained precession formulae, in terms of the difference in the eccentricity dependence $(1 - e^2)^3$ [21, 22, 18] and $\sqrt{1 - e^2}$ [23, 24]. The contribution of cosmological constant on the orbital dynamics is also investigated in other papers [25, 26, 27, 28, 29, 30, 31, 32, 33, 34], and constraints on the cosmological constant from the Sun's motion through the Milky Way is recently discussed [35].

In fact, this issue has been resolved in [36] using the general framework of radial/central force perturbation. Subsequently, it was shown that the physical interpretation of the formula derived in [36] is the precession of Hamilton's vector [37]. A similar problem as that in [36] was treated in terms of the Gaussian planetary equation [38], and Multiple Scales Method [39]. In [36, 38, 39], it was shown that the correct eccentricity dependence is $\sqrt{1 - e^2}$ as previously shown in [23, 24]. However, in spite of these facts, some confusion still seems to exist (Refer to e.g., [18]). Therefore, the purpose of this short note is to arrange the discussions on the perihelion/periastron advance due to the cosmological constant Λ .

2 Perihelion/periastron advance due to cosmological constant

2.1 Analytical approach by direct integration of perturbation potential/force

In order to examine the additional perihelion/periastron advance due to Λ , let us assume the spacetime to be spherically symmetric; without loss of generality, we write the metric as

$$ds^2 = - \left[1 - \frac{r_g}{r} + W(r) \right] c^2 dt^2 + \left[1 - \frac{r_g}{r} + W(r) \right]^{-1} dr^2 + r^2 d\Omega^2, \quad (1)$$

where $r_g \equiv 2GM/c^2$, $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ and $W(r)$ is an additional term that is a function of r and generally depends on the gravitational theories. Then, the geodesic equation of the test particle with mass m becomes

$$\left(\frac{dr}{d\tau} \right)^2 + U(r) = \left(\frac{E}{mc} \right)^2, \quad (2)$$

$$U(r) = c^2 \left(1 + \frac{L^2}{c^2 m^2 r^2} \right) \left[1 - \frac{r_g}{r} + W(r) \right]. \quad (3)$$

Here, τ is the proper time, and E and L are two constants of motion: the total energy and angular momentum, respectively. According to the standard approach, we define the Newtonian potential $\Phi(r)$ as

$$\Phi(r) \equiv \frac{1}{2} \lim_{c^2 \rightarrow \infty} [U(r) - c^2] = -\frac{GM}{r} + \frac{\mathcal{L}^2}{2r^2} + \frac{1}{2}W(r)c^2, \quad (4)$$

where we set $\mathcal{L} = L/m$ and let $\mathcal{E} = (1/2)(E/mc)^2$; then, we have,

$$\frac{1}{2} \left(\frac{dr}{d\tau} \right)^2 + \Phi(r) = \mathcal{E}. \quad (5)$$

If we rewrite $V(r) \equiv (1/2)W(r)c^2$ and $\tau \approx t$ (t is coordinate time), Eq. (5) becomes completely equivalent to Eq. (15) of [36]. Thus, henceforth, we follow the approach employed in [36].

The perihelion/periastron shift $\Delta\theta_p$ due to the perturbation potential $V(r)$ with respect to Newtonian potential $-GM/r$ is expressed as (see Eq. (30) in [36])

$$\Delta\theta_p \equiv \frac{-2\ell}{GM e^2} \int_{-1}^1 \frac{z dz}{\sqrt{1-z^2}} \frac{dV(z)}{dz}, \quad (6)$$

in which $r = \ell/(1+ez)$ and $\ell = \mathcal{L}^2/(GM) = a(1-e^2)$, a being the semi-major axis of the orbit. It should be noted that in [36], $V(r)$, or equivalently $V(z)$, is defined including the mass of the orbiting particle m (refer to Eq. (15) in [36]). However, in our case, $V(r)$ or $V(z)$ does not contain m ; hence, hereafter, m does not appear in the precession formula.

We are interested in the power-law perturbing potentials of the type $V(r) = \alpha_n r^n$. Noting the relation,

$$V(r) = -\alpha_{-(n+1)} r^{-(n+1)} = -\alpha_{-(n+1)} (1+ez)^{n+1} / \ell^{n+1}, \quad (7)$$

we have the precession formula,

$$\Delta_p(-(n+1)) = -\frac{2\alpha_{-(n+1)}(n+1)}{GM \ell^n e} \int_{-1}^1 \frac{z(1+ez)^n}{\sqrt{1-z^2}} dz. \quad (8)$$

For discussing the effect of cosmological constant Λ , the Schwarzschild–de Sitter or Kottler metric [40],

$$ds^2 = -\left(1 - \frac{r_g}{r} - \frac{\Lambda}{3}r^2\right) c^2 dt^2 + \left(1 - \frac{r_g}{r} - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2 d\Omega^2 \quad (9)$$

is widely used and r_g/r term causes famous general relativistic perihelion/periastron shift. In the case of perihelion/periastron advance due to Λ , we consider $W(r) = -(1/3)\Lambda r^2$ and

then $V(r) = -(1/6)\Lambda c^2 r^2$. It should be noted in this case, $-(n+1) = 2$; thus, $n = -3$ and $\alpha_2 = -(1/6)\Lambda c^2$.

The integral part in Eq. (8) is

$$\int_{-1}^1 \frac{z(1+ez)^{-3}}{\sqrt{1-z^2}} dz = -\frac{3\pi}{2} \frac{e\sqrt{1-e^2}}{(1-e^2)^3}. \quad (10)$$

Therefore, from Eqs. (8) and (10), the perihelion/periastron advance due to Λ is given by the following equation (again, note $\ell = a(1-e^2)$ and $n = -3$)

$$\Delta\omega_\Lambda = \frac{\pi c^2 \Lambda a^3}{GM} \sqrt{1-e^2}. \quad (11)$$

Eq. (11) is in agreement with the results of [23, 24, 38].

Let us summarize the outline of above derivation. We start from the SdS/Kottler metric Eq. (9) and its “Newtonian approximation (weak field approximation)”. Then the perturbation potential $U_\Lambda = -(1/6)\Lambda r^2$ or perturbation force $F_\Lambda = (1/3)\Lambda r$ is determined. The result, Eq. (11), is obtained exactly by “the direct integration of given perturbation potential or force” e.g. Eq. (10), with the help of the knowledge on Gaussian hypergeometric function ${}_2F_1(\alpha, \beta, \gamma; x)$, see Eq. (37) in [36]. We mention that Eq. (10) can be shortly calculated by the variable transformation,

$$z = \frac{\cos \xi - e}{1 - e \cos \xi}, \quad (12)$$

which produces the integral

$$\frac{\sqrt{1-e^2}}{(1-e^2)^3} \int_0^\pi (\cos \xi - e)(1 - e \cos \xi) d\xi = -\frac{3\pi}{2} \frac{e\sqrt{1-e^2}}{(1-e^2)^3}. \quad (13)$$

It is worthy to note when integrating Eq. (10), the integral part is not expanded with respect to the orbital eccentricity e .

2.2 Standard perturbation method

The perihelion/periastron advance due to Λ is considered by means of the standard perturbation method. Here let us discuss this approach according to [22].

Adopting SdS/Kottler metric Eq. (9), we begin with the second-order geodesic equation for time-like world-line,

$$\frac{d^2 u}{d\phi^2} + u - \frac{GM}{L_z^2} = \frac{3GM}{c^2} u^2 - \frac{\Lambda c^2}{3L_z^2 u^3}, \quad u = \frac{1}{r}, \quad (14)$$

here L_z is the z component of orbital angular momentum¹, and right-hand side of (14) can be considered as the perturbation to the Keplerian motion. The Keplerian motion is described by

$$\frac{d^2 u}{d\phi^2} + u - \frac{GM}{L_z^2} = 0 \quad (15)$$

and its solution is

$$\frac{1}{r} = u = \frac{GM}{L_z^2} (1 + e \cos \phi). \quad (16)$$

Since it is well-known that the first term in right-hand side of Eq. (14) causes famous general relativistic precession formula, $\Delta \omega_{\text{gr}} = 6\pi GM / (ac^2(1 - e^2))$, and Eq. (14) is linear differential equation, then we concentrate on the perturbation due to the cosmological constant Λ ;

$$\frac{d^2 u}{d\phi^2} + u - \frac{GM}{L_z^2} = -\frac{\Lambda c^2}{3L_z^2 u^3}. \quad (17)$$

Inserting Eq. (16) into right-hand side of Eq. (17), we have

$$-\frac{\Lambda c^2}{3L_z^2 u^3} = -\frac{\Lambda c^2 L_z^4}{3(GM)^3} \frac{1}{(1 + e \cos \phi)^3}. \quad (18)$$

Here, it is worthy to mention when obtaining Eq. (11), the part $1/(1 + e \cos \phi)^3$ is exactly integrated with changing the variable $\cos \phi$ with z (see Eq. (10)). However, in the standard literature such as [22], this part is expanded with respect to the orbital eccentricity e ,

$$\frac{1}{(1 + e \cos \phi)^3} = 1 - 3e \cos \phi + 6e^2 \cos^2 \phi + \mathcal{O}(e^3), \quad (19)$$

and only $e \cos \phi$ term is retained in [22]. Comparing with the derivation process of general relativistic precession formula, the precession formula due to Λ is derived as (see Eq. (14.25) in [22])

$$\Delta \omega_{\Lambda} \approx \frac{\pi c^2 \Lambda L_z^6}{(GM)^4}. \quad (20)$$

It should be emphasized here although the expansion itself is “truncated at the first order in e ”, the orbital angular momentum L_z is given by

$$L_z = \sqrt{GM\ell} = \sqrt{GMa(1 - e^2)} \quad (21)$$

then the quadratic terms in orbital eccentricity e is remained in the final result due to L_z but these terms are not cared in the lowest order perturbation method considered.

¹ In [22], the z component of angular momentum is expressed by h instead of L_z .

Inserting Eq. (21) into Eq. (20), it is found

$$\Delta\omega_\Lambda = \frac{\pi c^2 \Lambda a^3}{GM} (1 - e^2)^3, \quad (22)$$

which is also same results of [21, 18].

If we expand the integral part in Eq. (10) up to the second order in e and integrate, then we have

$$\int_{-1}^1 \frac{z(1+ez)^{-3}}{\sqrt{1-z^2}} dz \simeq \int_{-1}^1 \frac{z(1-3ez+6e^2z^2)}{\sqrt{1-z^2}} dz = -\frac{3\pi}{2}e, \quad (23)$$

and combining with Eq. (8), Eq. (22) is recovered.

The eccentricity dependence of Eqs. (11) and (22) is plotted in Fig. 1. It is clear that Eqs. (11) and (22) behave quite differently as the orbital eccentricity becomes large.

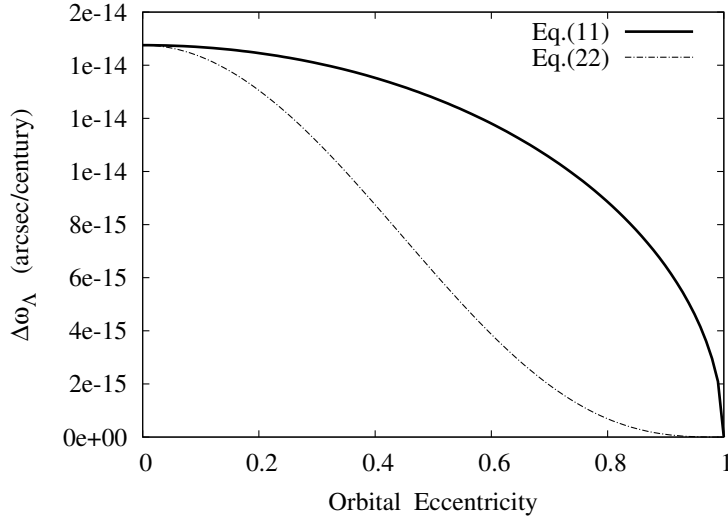


Fig. 1 The eccentricity dependence of Eqs. (11) and (22). In this plot, we set $\Lambda \approx 10^{-52} \text{ m}^{-2}$, $M = 2.0 \times 10^{30} \text{ kg}$, $a = 1.5 \times 10^{11} \text{ m}$.

3 Summary

In this short note, we commented on the perihelion/periastron advance of celestial bodies due to the cosmological constant Λ . As we stated before, our prescription is based on [36], in which the problem of perihelion/periastron shift due to the cosmological constant Λ has

practically been resolved. However, we hope that this note contributes to clear up the confusion about the perihelion/periastron advance due to the cosmological constant Λ , and is helpful in discussing the probability of alternative theories of gravitation since Λ -like terms arise also in various theoretical contexts such as $f(T)$ [41] and $f(R)$ gravity [42].

Just before closing this letter, we would mention the following issue; for high orbital eccentricity e and large central mass M , the general relativistic value of perihelion/periastron advance becomes large, while Eqs. (11) and (22) tend to zero (see again Fig. 1). This property in Eqs. (11) and (22) seems to be physically counter-intuitive. As a possibility, this counter-intuitive feature may have roots in the fact that essentially, Schwarzschild–de Sitter/Kottler metric does not become asymptotically flat spacetime. However the derivation of Eqs. (11) and (22) may assume implicitly that the background metric is Minkowskian. Therefore, the problem on perihelion/periastron advance due to cosmological constant Λ should be investigated and discussed further in a careful manner in the context of real expanding universe (see, for example, [43], and similar topic was recently discussed by [44]).

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